Lesson Plan: Flexagons & Topology

Background

Scientific discoveries are often sitting in plain sight, waiting to be discovered! This was the case for British mathematician Arthur H. Stone in 1939. When attempting to fit his large American paper into his small European binder (see **Note 1**), he cut off the overlapping strips of paper so that the paper would fit. As a result, he was left with a large pile of long, narrow pieces of paper. So, what did he do? He began folding them. Eventually, he folded a long strip of paper into a **hexagon** – a shape with six sides and six angles – but one that could be folded in on itself and refolded, or **flexed**, to expose different sides or **faces** of the paper. This became known as the **hexaflexagon**, a specific shape of **flexagon**, which is a flat folded paper object with hidden faces that appear when the paper is refolded (**Note 2**).

The hexaflexagon is a simple model used in explaining **topology**, a term that describes the shape, size, or other surface features of objects. For example, the spatial relationships of objects – how objects and their components are positioned relative to one another – are topological features. Topology in mathematics refers to the study of connections in shapes or spaces. A famous problem in topology is called the Seven Bridges of Königsberg, which asks the reader to find a route through the city of Königsberg that crosses each of the city's seven bridges once, but only once. The solution to this problem by the Swiss mathematician, Euler, led to the development of the branch of mathematics called **graph theory**, which is the study of connections between objects or points.

Hexaflexagons provide an interesting and fascinating way to become acquainted with basic concepts of topology. A flat two-dimensional **hexagon** is expected to have six sides and two faces, but the standard hexaflexagon has *three faces*. If you can't imagine a flat two-dimensional hexagon with three faces without seeing one, try making one yourself! Depending on the number of folds, different classes of flexagons can have *three or more* faces (**Note 2**)!

Understanding topology will give you the ability to understand much broader concepts in mathematics and physics that build on properties of three-dimensional space. Understanding the ways in which an object can exist within the planes of three-dimensional space opens possibilities of understanding dimensions beyond the third dimension. Dimensions beyond the third dimension are necessary to understand the relationship between space and time, and time travel!

Objectives & Grade Level

Students will learn how a hexagon is related to a hexaflexagon and how to create a simple, three-faced hexaflexagon. They will also learn what a Möbius strip is and how a Möbius strip is related to a hexaflexagon. This will enable them to understand a basic concept of topology and further their understanding of complex shapes. Appropriate for middle to high school classes.

Materials

- Paper
- Scissors
- Markers/colored pencils
- Tape

Procedure

1. Cutting the paper

First, you will need a narrow strip of paper that can be folded into nine triangles with equal sides. Take a piece of paper (either US letter size 8.5 x 11 inch paper or A4 21 x 29.7 cm paper) and make a fold of approximately 1.25 inches (or 3 cm) along the long side. Alternatively, you can print the templates at the end of this lesson plan and cut out the paper strips.

If you are using American college-ruled paper, you can use the red line that is printed vertically along one edge of the paper to fold your paper (**Figure 1A**, red arrow). Then use scissors to cut along the fold that you made. You should have a narrow strip of paper (**Figure 1B**).

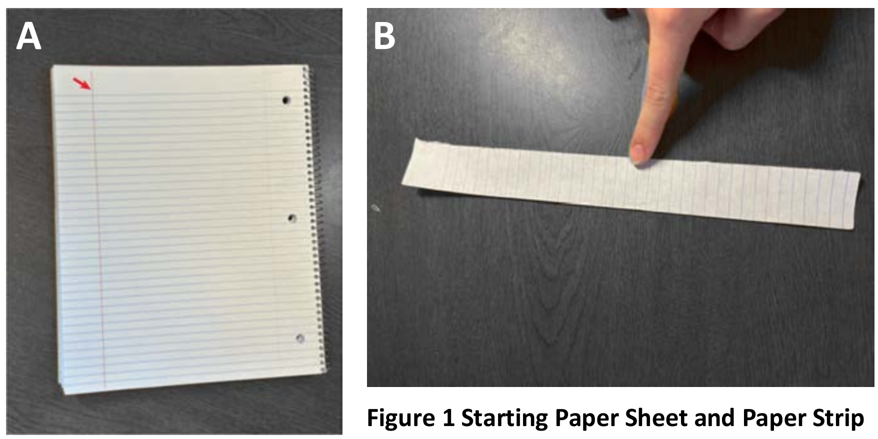


Figure 1 Starting Paper Sheet and Paper Strip

2. Marking the triangles

At one end of the paper strip, make the first fold for a triangle with equal sides (**Figure 2A**, green dashed lines). This will leave a small overhang. Don't worry – you can cut it off the ends later! Mark nine triangles of the same size (**Figure 2B**, red numbers). The extra triangles and edges at the ends of the strip (**Figure 2B**, red dashed lines) can be cut off the strip after you finish marking the triangles.

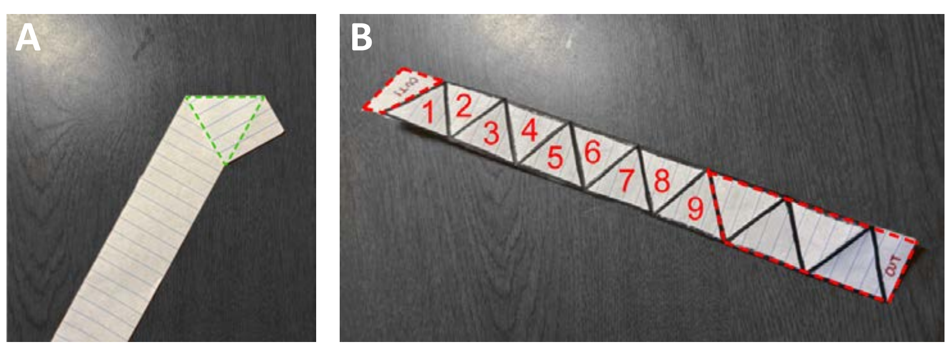
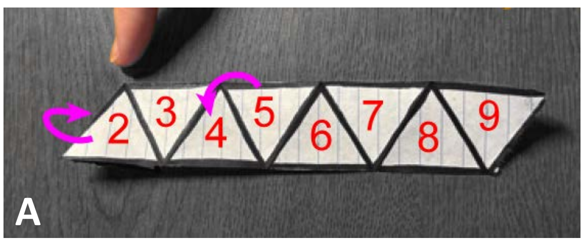


Figure 2 Marking Nine Triangles

3. **Folding the hexaflexagon**

After marking the triangles, make a crease along each line that forms a triangle edge. Then lay the strip flat and make the following three folds: i) First fold triangle 1 backward onto triangle 2 so the side with number 2 is facing you and number 1 is facing down (**Fig. 3A**). ii) Then fold triangle 5 onto triangle 4, so the numbers 4 and 5 face each other. Triangle 5 will be facing down so you cannot see the number 5 (**Fig. 3B**). Triangles 6, 7, 8 and 9 will also be facing down. iii) Then fold triangle 8 backward onto triangle 7 (**Fig. 3B**), so triangle 8 is facing up and triangle 7 is facing down (**Fig. 3C**). The shape should be a hexagon with triangles 2 ,3, 8 and 9 facing up. Finally, on the bottom, tape the back edge of triangle 9 to the front edge of triangle 1, which is underneath triangle 2. The tape should be on the bottom of the hexaflexagon and triangles 1, 6 and 7 will be facing down.



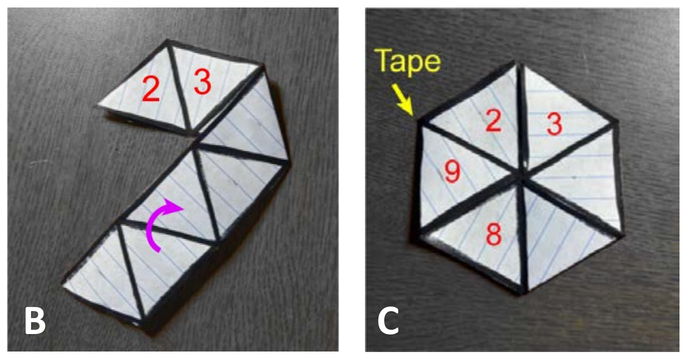


Figure 3 Folding the Hexaflexagon. A) Fold triangle 1 backward and triangles 4 and 5 together. **B**) Then fold triangle 8 backward. **C**) On the bottom, tape the back of triangle 9 to the front of triangle 1.

4. **Coloring the hexaflexagon**

Now comes the fun part! Using your color pencils or markers, color the top face of your hexaflexagon one color and the bottom of your hexaflexagon a different color (**Figure 4 A,B**).

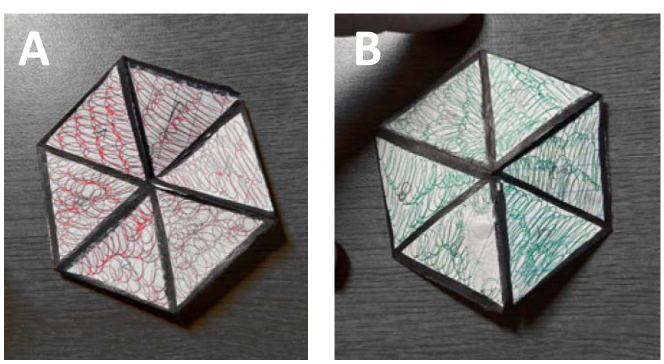


Figure 4 Hexaflexagon Faces. **A**) Top and **B**) Bottom.

5. **Flexing your hexaflexagon**

Now we will show you how to flex your hexaflexagon and find the hidden third face. Carefully push inward along the creases that you made earlier to make your hexaflexagon form three points (**Figure 5A**). **Note**: you should push inward along the creases between triangles that are continuous, not where the paper folds over itself. These are the creases between the numbered sides of triangles 2 and 3, and triangles 8 and 9, and the backs of triangles 4 and 5.

You should see an opening in the center of your hexaflexagon (**Figure 5B**). Pull open the hexaflexagon at the opening to find a new face, one that you haven't colored yet! This is the hidden face – it shows the numbered sides of triangles 4 and 5 (**Figure 5B**).

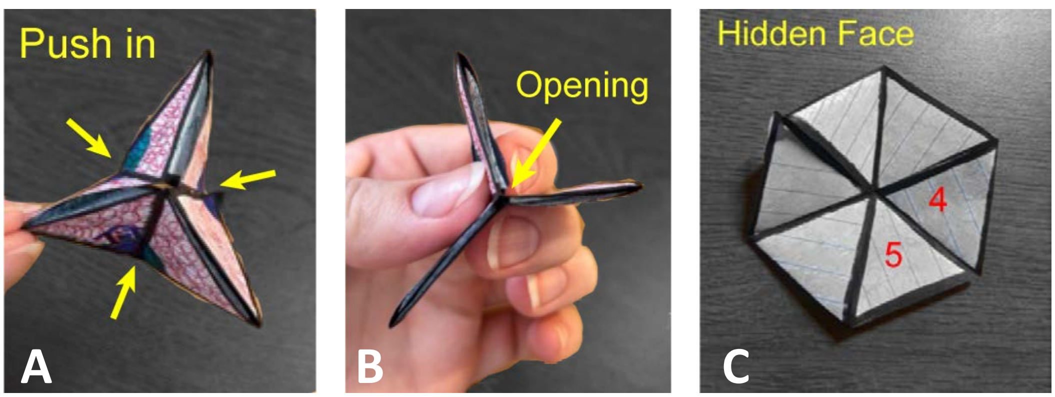


Figure 5 Flexing the Hexaflexagon. A) Push in along the three continuous sides of the hexaflexagon. **B**) Then find the opening and pull open the hexaflexagon. **C**) You will find the hidden face, which you have not colored yet.

You have discovered a new face! Color the new face a different color. Then push inward on the creases again – these are the creases between the numbered sides of triangles 4 and 5, and the backs of 8 and 9, together with the backs of triangles 1 and 3. What do you see when you open the hexaflexagon? You should see the face of the hexaflexagon that was facing downward when you started! To check this, flip the hexaflexagon over and look at the triangle numbers that were on the bottom – they should be the ones that were on the top when you first folded the hexaflexagon.

6. **Flex and fold your hexaflexagon!**

Have fun! Experiment to discover other features of hexaflexagons! How is your new hexaflexagon related to a hexagon? Remember that a hexagon has six sides and two faces, but your hexaflexagon has six sides and three faces! Not only are there three different faces to an object that seems to only have two, but the triangles also appear in different orientations on each face. We made a hexaflexagon with a dinosaur so you can see how the triangles change in orientation on different faces. On one face of the hexaflexagon, the T-Rex is headed straight towards the house (**Figure 6A**). But after a quick refolding, the dinosaur and the house are on two completely different planets (**Figure 6B**)!

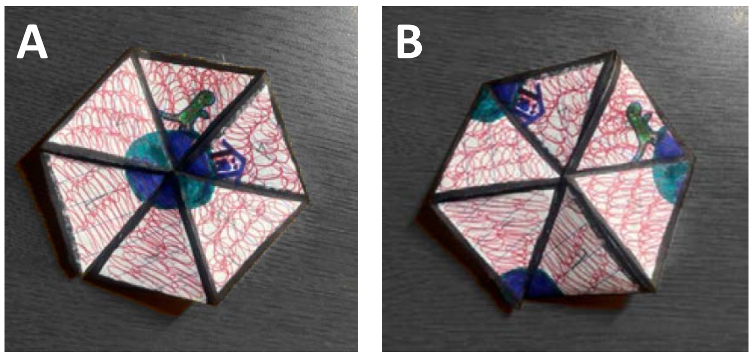


Figure 6 Hexaflexagon Triangles Reorient upon Refolding

Notes

1. **Paper size:** Writing paper in the United Kingdom, where Arthur H. Stone was from, used Imperial Paper sizes until the 1970s when they were replaced by the current A Series paper sizes. There were different Imperial Paper sizes that fell into a broad range of sizes (4 - 8 inches wide x 6 - 10 inches high), all of which were smaller than the US Standard Letter size (8.5 inches wide x 11 inches high), which was most likely the size that Stone was trying to fit into his notebook.

2. **Flexagon names:** The standard hexaflexagon is a **tri**hexaflexagon – this means that it is a six-sided (hexagonal) shape with three (**tri-**) different sides or faces. The prefixes tetra-, penta- and octa- similarly refer to the number of faces for the hexaflexagon, either four (teta-), five (penta-) or eight (octa-). There are many more, as the number of faces can vary, depending on whether the starting paper strip is straight or not, and the folding pattern that you use! You can also make flexagons in different shapes of squares, pentagons or octagons!

3. **The flexagon and the Möbius strip:** While flexagons and Möbius strips may appear at first to be completely different, a flexagon is actually a **Möbius strip**! What is a Möbius strip? It is a surface that has no ends. You can make a Möbius strip that has one continuous surface using a strip of paper. By adding a twist in the strip of

paper before you tape the ends together, the surface that originally had two surfaces now has only one continuous surface (**Figure 7**).

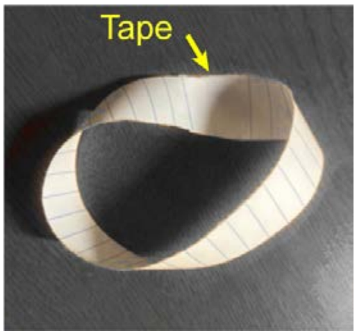


Figure 7 Möbius Strip

One way to understand this is by imagining an ant walking along the surface of a piece of paper. If the strip of paper is taped together to form a loop without a twist in the paper, the ant would walk either on the inside or the outside of the paper. By contrast, on a Möbius strip, the ant walks on the inside and outside of the paper at the same time! The inside and outside of the paper strip are the same for a Möbius strip!

So how does this relate to flexagons? Flexagons are just Möbius strips with twists in the paper strip that forms a loop. They are harder to recognize because of they are folded into a two-dimensional shape. Because of the way the flexagon is folded, there are twists in the paper strip when you tape the ends together. To prove this to yourself, try gently pulling your hexaflexagon apart! You should be able to see the paper form a loop with twists in it. To learn more about flexagons as Möbius strips, go the web site below:
https://www.futurelearn.com/info/courses/flexagons/0/steps/48223

4. **Advanced topic 1:** If you want to know the number of twists a flexagon has, you can use the following equation to calculate the number of half twists. **Note**: a half twist is 180°, which makes the strip turn and face in the opposite direction.

   Half twists = 3N - n, where N= number of faces on the flexagon, and n= number of triangles on each face.

   Thus, for our hexaflexagon above, the calculation would be 3(3 faces) - 6 triangles on each face = 3 half twists in the hexaflexagon.

5. **Advanced topic 2:** The **Tuckerman transverse** is a diagram created by Bryant Tuckerman, a mathematician, to show the order of appearance of the flexagon faces, i.e., the face that is expected to be outward at a given fold. For the trihexaflexagon, like the one you just made using this lesson plan, the Tuckerman transverse is a simple triangle with arrows showing the order in which the faces will appear. However, as the flexagon becomes more complex, the transverse will become more complex. To learn more about the Tuckerman transverse, go to the web sites below:
   http://loki3.com/flex/cycles-traverses.html
   https://www.futurelearn.com/info/courses/flexagons/0/steps/48214

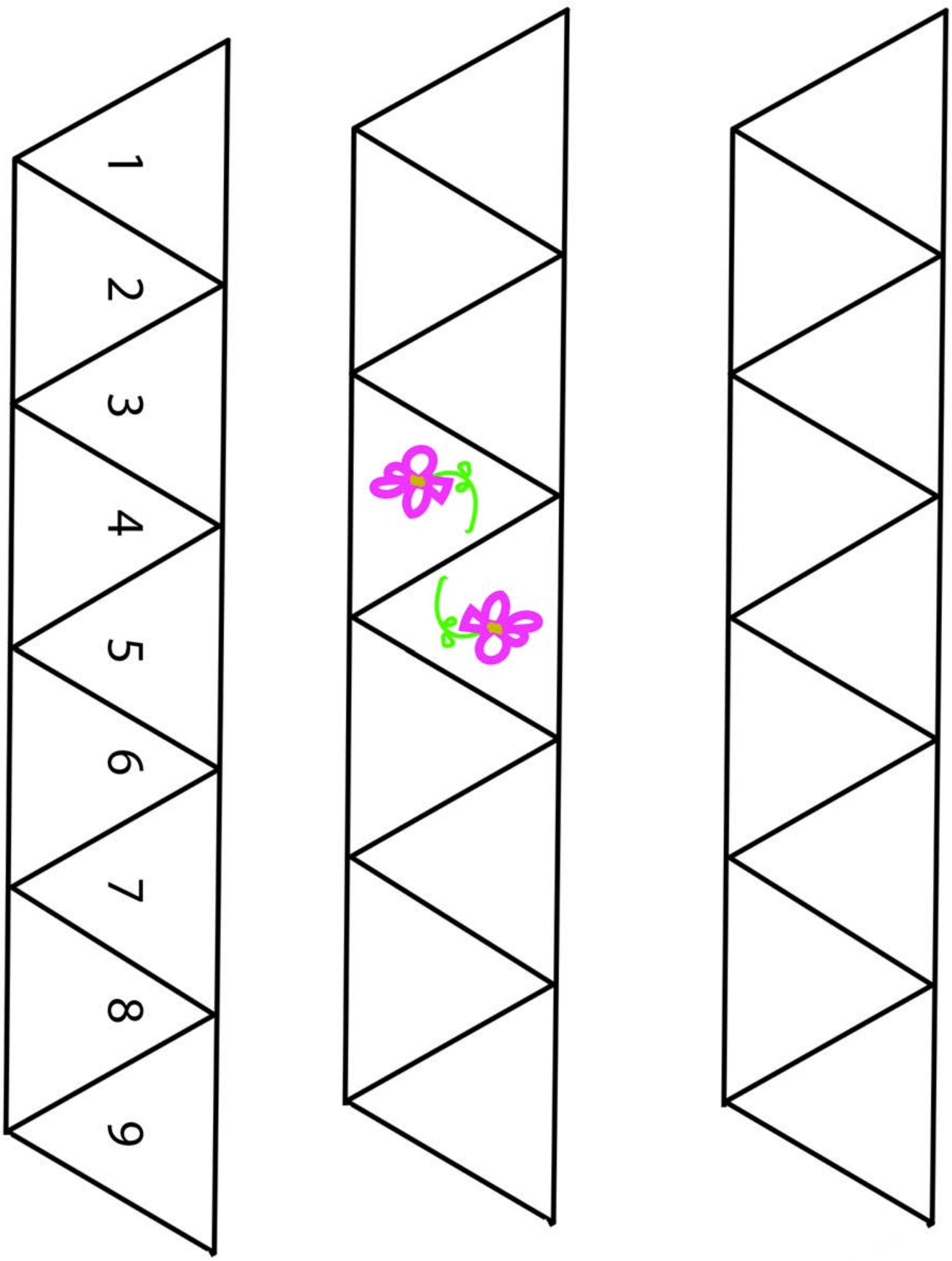
6. **Flexagons and chirality:** Chirality refers to the relative position of the features of an object with respect to one another. In terms of flexagons, chirality tells us the order in which the flexagon faces will appear according to the Tuckerman traverse (see **Note 5 Advanced topic 2** above). When refolding the flexagon sequentially, the different faces will appear in a given order. However, turning the flexagon upside down and refolding the

flexagon sequentially will result in the opposite rotation of the faces. In other words, the Tuckerman traverse is reversed when the flexagon is turned upside down! This means that the chirality of the hexaflexagon is reversed when you flex the hexaflexagon from the bottom, compared to the top. Try this using your hexaflexagon with different colored faces!

Chirality is an important concept to understand for those interested in fields that require knowing the relationship of objects or their features with respect to one another. An example that you might not have expected is the field of chemistry. Chirality in chemistry is used to describe the arrangement of side groups on molecules, which give molecules unique chemical properties. To learn more about chirality, go to the site below:

https://en.wikipedia.org/wiki/Chirality

7. **Hexaflexagon templates:** The three templates on the following page have nine equilateral triangles drawn on each strip and each strip can be used to make a trihexaflexagon. One template is numbered to make it easier for you to fold your first hexaflexagon. The second template has a flower on two triangles, which should appear on the hidden face, if you fold the hexaflexagon using the instructions in step 3 above. The third template is blank so that you can make your own design after you fold the hexaflexagon.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Resources

Web sites

Wikipedia. Flexagon. https://en.wikipedia.org/wiki/Flexagon
Wikipedia. Topology. https://en.wikipedia.org/wiki/Topology

Books

Oakley, C. O. and R. J. Wisner. 1957 Flexagons. *American Mathematical Monthly* 64:143–154. https://doi.org/10.2307/2310544

Gardner, M. 1988 *Hexaflexagons and Other Mathematical Diversions: The First Scientific American Book of Mathematical Puzzles and Games.* University of Chicago Press.

Mitchell, D. 2008 *The Magic of Flexagons: Manipulative Paper Puzzles to Cut Out and Glue Together.* AmazonUs/INDPB.

Pook, L. 2017 *Flexagons Inside Out.* Racehorse for Young Readers.

Videos

https://www.youtube.com/watch?v=IgDRTPxAl1k

https://www.youtube.com/watch?v=VIVIegSt81k

https://www.youtube.com/embed/AmN0YyaTD60

https://www.youtube.com/watch?v=paQ10POrZh8

Acknowledgements

We thank Lauren Page for insightful and valuable comments that we used to improve the presentation of this lesson plan. The original background of photos used in the figures of this lesson plan was brown, but was replaced digitally by the dark gray-black background for better contrast.

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